Cluster automorphism groups and automorphism groups of exchange graphs [1]

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Abstract

For a coefficient free cluster algebra $\mathcal{A}$, we study the cluster automorphism group $Aut(\mathcal{A})$ and the automorphism group $Aut(E_{\mathcal{A}})$ of its exchange graph $E_{\mathcal{A}}$. We show that these two groups are isomorphic with each other, if $\mathcal{A}$ is of finite type, excepting types of rank two and type F_4, or $\mathcal{A}$ is of skew-symmetric finite mutation type.

Key words. Cluster algebras; Exchange graphs; Cluster automorphism groups.

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1 Introduction

Cluster algebras are introduced by Sergey Fomin and Andrei Zelevinsky in [9]. In this paper we consider a special class so called coefficients free cluster algebras of geometric type, which is a commutative $\mathbb{Z}$-algebra defined through a skew-symmetrizable square matrix. Starting from a seed, which is a pair consisting of a set (cluster) of n indeterminate elements and a skew-symmetrizable square matrix of rank n, we get a new seed by a mutation, that makes a birational transform on the cluster and performs an operation upon the matrix. Then as a $\mathbb{Z}$-subalgebra of the rational function field over the initial cluster, the cluster algebra is generated by the elements (cluster variables) in the clusters obtained by recursively mutations. A cluster algebra has nice combinatorical structures which are given by mutations, and these structures are captured by an exchange graph, which is a graph with seeds as vertices and with mutations as edges.

If there are finite clusters in a cluster algebra, then we say that it is of finite type. These cluster algebras are classified in [10], which corresponds to the Killing-Cartan classification of complex semisimple Lie algebras, equivalently, corresponds to the classification of root systems in Euclidean space. If there are finite matrix classes in the seeds of a cluster algebra, then we call it a cluster algebra of finite mutation type, where two matrices are in a same class if one of them can be obtained from the other matrix by simultaneous relabeling of the rows and columns. The cluster algebras of finite mutation type defined by skew-symmetric matrices are classified in [8], a large class of them arises from oriented marked Riemann surfaces [6], and there are 11 exceptional ones. The classification of skew-symmetrizable cluster algebras of finite mutation type is given in [7] via operations so called unfoldings upon the skew-symmetric cluster algebras of finite mutation type.

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A cluster automorphism of a cluster algebra is a permutation of the cluster variable set, which commutates with mutations. Cluster automorphisms naturally consist a group: cluster automorphism group. This group reveals the symmetries of the cluster algebra, especially the symmetries of combinatorical structures and algebra structures. This group is introduced in [1] for a coefficient free cluster algebra, and in [4] for a cluster algebra with coefficients. An automorphism of the exchange graph of a cluster algebra is firstly studied in [4], it is an automorphism of a graph. All of the automorphisms consist a group, which describes symmetries of the exchange graph, in other words, describes symmetries of the combinatorical structures of the cluster algebra.

In this paper, for a coefficient free cluster algebra $\mathcal{A}$ with exchange graph $E_{\mathcal{A}}$, we consider the relations between the cluster automorphism group $Aut(\mathcal{A})$ and the automorphism group $Aut(E_{\mathcal{A}})$. Generally, $Aut(\mathcal{A})$ is a subgroup of $Aut(E_{\mathcal{A}})$, and may be a proper subgroup, for example see Example 2, and Example 4. The main result of this paper is that these two groups are isomorphic with each other, if $\mathcal{A}$ is of finite type, excepting types of rank two and type F_4 (Theorem 3.7), or $\mathcal{A}$ is of skew-symmetric finite mutation type (Theorem 3.8). Therefore in some degree, for these cluster algebras, the algebra structures are also captured by the exchange graphs.

To prove these results, we describe $E_{\mathcal{A}}$ more precisely. In subsection 3.1, we define layers of geodesic loops of $E_{\mathcal{A}}$ by using the distance of a vertex to a fixed vertex on $E_{\mathcal{A}}$. An easy observation is that an isomorphism of exchange graphs should maintain the combinatorial numbers of the layers of geodesic loops based on the corresponding vertices (Remark 3.2(4)). Then by this observation, we directly show in Examples 5, 6, 8, 9 that for a cluster algebra of type A_3, B_3, C_3, $\tilde{A}_2$ or T_3 (the cluster algebra from an once punctured torus), we have $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$. For the general cases we reduce them to above five cases (Theorems 3.7, 3.8).

The paper is organized as follows: we recall preliminaries on cluster algebras, cluster algebras of finite mutation type and cluster automorphisms in section 2, then we prove the main Theorems in section 3.

2 Preliminaries

2.1 Cluster algebras

We recall basic definitions and properties on cluster algebras in this subsection.

Definition 2.1. *[9](Labeled seeds). A* labeled seed *is a pair $\Sigma = (\mathbf{x}, B)$, where*

- $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ *is a set with n elements;*
- $B = (b_{x_j x_i})_{n\times n} \in M_{n\times n}(\mathbb{Z})$ *is a matrix labeled by $\mathbf{x}\times\mathbf{x}$, and it is skew-symmetrizable, that is, there exists a diagonal matrix D with positive integer entries such that DB is skew-symmetric.*

The set $\mathbf{x}$ is the *cluster* of Σ, B is the *exchange matrix* of Σ. We also write b_{ji} to an element $b_{x_j x_i}$ in B for brevity. The elements in $\mathbf{x}$ are the *cluster variables* of Σ. We always assume through the paper that B is indecomposable, that is, for any $1 \leqslant i, j \leqslant n$, there is a sequence $i_0 = i, i_1, \cdots, i_m, i_{m+1} = j$, such that $b_{i_k, i_{k+1}} \neq 0$ for any $0 \leqslant k \leqslant m$. Given a cluster variable x_k, we produce a new labeled seed by a mutation. We also assume that $n > 1$ for convenience.

Definition 2.2. *[9](Seed mutations). The labeled seed $\mu_k(\Sigma) = (\mu_k(\mathbf{x}), \mu_k(B))$ obtained by the* mutation *of Σ in the direction k is given by:*

- $\mu_k(\mathbf{x}) = (\mathbf{x} \setminus \{x_k\}) \sqcup \{x'_k\}$ *where*

$$x_k x'_k = \prod_{\substack{1\leqslant j\leqslant n\ ;\\ b_{jk}>0}} x_j^{b_{jk}} + \prod_{\substack{1\leqslant j\leqslant n\ ;\\ b_{jk}<0}} x_j^{-b_{jk}}.$$

- $\mu_k(B) = (b'_{ji})_{n\times n} \in M_{n\times n}(\mathbb{Z})$ *is given by*

$$b'_{ji} = \begin{cases} -b_{ji} & \text{if } i = k \text{ or } j = k\ ; \\ b_{ji} + \frac{1}{2}(|b_{ji}|b_{ik} + b_{ji}|b_{ik}|) & \text{otherwise.} \end{cases}$$

It is easy to check that a mutation is an involution, that is $\mu_k\mu_k(\Sigma) = \Sigma$.

Definition 2.3. *[11](n-regular patterns). A n-regular tree $\mathbb{T}_n$ is diagram, whose edges are labeled by $1, 2, \cdots, n$, such that the n edges emanating from each vertex receive different labels. A* cluster pattern *is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, so that the labeled seeds assigned to the endpoints of any edge labeled by k are obtained from each other by the seed mutation in direction k. The elements of Σ_t are written as follows:*

$$\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}), \quad B_t = (b^t_{ij}). \tag{1}$$

Note that $\mathbb{T}$ is in fact determined by any fixed labeled seed on it.

Now we are ready to define cluster algebras.

Definition 2.4. *[11](Cluster algebras). Given a seed Σ and a cluster pattern $\mathbb{T}_n$ associated to it, we denote*

$$\mathscr{X} = \bigcup_{t\in\mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} \ :\ t \in \mathbb{T}_n\ ,\ 1 \le i \le n\}\ , \tag{2}$$

the union of clusters of all the seeds in the pattern. We call the elements $x_{i,t} \in \mathscr{X}$ the cluster variables. *The* cluster algebra *$\mathcal{A}$ associated with Σ is the $\mathbb{Z}$-subalgebra of the rational function field $\mathcal{F} = \mathbb{Q}(x_1, x_2, \cdots, x_n)$, generated by all cluster variables: $\mathcal{A} = \mathbb{Z}[\mathscr{X}]$.*

For a skew-symmetrizable matrix $B = (b_{ji})_{n\times n}$, one can associate it to an valued quiver (quiver for brevity) $Q = (Q_0, Q_1, \upsilon)$ as follows: $Q_0 = \{1, 2, \cdots, n\}$ is a set of vertices; for any two vertices j and i, if $b_{ji} > 0$, then there is an arrow α from j to i, these arrows form the set Q_1; for an arrow α from j to i, we assign it a pair of values: $(\upsilon_1(\alpha), \upsilon_2(\alpha)) = (b_{ji}, -b_{ij})$. Since B is an indecomposable skew-symmetrizable matrix, the defined valued quiver Q is connected and there is no loops nor 2-cycles in Q. Then we can define a mutation of the valued quiver by the mutation of the matrix, we refer to [13, 3] for details. We say two quivers Q and Q' mutation equivalent, if the corresponding matrices are mutation equivalent, that is, one of them can be obtained from the other one by a finite sequence of mutations. We also write $(\mathbf{x}, Q)$ to the labeled seed $(\mathbf{x}, B)$, and write $\mathcal{A}_Q$ to the cluster algebra defined by Σ. We say that the quiver and the defined cluster algebra are skew-symmetric, if the corresponding matrix is skew-symmetric. If the cluster algebra is of finite type [10](see in section 2.2 for cluster algebras of finite type) or of skew-symmetric type [12], then a cluster determines the quiver, and we denote the quiver of a cluster $\mathbf{x}$ by $Q(\mathbf{x})$.

Example 1. *Let B be the following matrix, it is a skew-symmetrizable matrix with diagonal matrix $D = diag\{2, 2, 1, 1\}$. The quiver corresponding to B is Q, where we always delete the trivial pairs of values $(1, 1)$, and replace a arrow assigning pair (m, m) by m arrows.*

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$$Q: \quad 1 \longrightarrow 2 \xleftarrow{(2,1)} 3 \Longrightarrow 4$$

Definition 2.5. *[11](Seeds) Given two labeled seeds $\Sigma = (\mathbf{x}, B)$ and $\Sigma' = (\mathbf{x}', B')$, we say that they define the same* seed *if Σ' is obtained from Σ by simultaneous relabeling of the sets $\mathbf{x}$ and the corresponding relabeling of the rows and columns of B.*

We denote by $[\Sigma]$ the seed represented by a labeled seed Σ. The cluster $\mathbf{x}$ of a seed $[\Sigma]$ is a unordered n-element set. For any $x \in \mathbf{x}$, there is a well-defined mutation $\mu_x([\Sigma]) = [\mu_k(\Sigma)]$ of $[\Sigma]$ at direction x, where $x = x_k$. For two same rank skew-symmetrizable matrices B and B', we say $B \cong B'$, if B' is obtained from B by simultaneous relabeling of the rows and columns of B. Then the exchange matrices in any two labeled seeds representing a same seed are isomorphic. The isomorphism of two exchange matrices induces an isomorphism of corresponding quivers. For convenience, in the rest of the paper, we also denote by Σ the seed $[\Sigma]$ represented by Σ.

Definition 2.6. *[11](Exchange graphs) The* exchange graph *of a cluster algebra is the n-regular graph whose vertices are the seeds of the cluster algebra and whose edges connect the seeds related by a single mutation. We denote by $E_{\mathcal{A}}$ the exchange graph of a cluster algebra $\mathcal{A}$.*

Clearly, the exchange graph of a cluster algebra is a quotient graph of the exchange pattern, its vertices are equivalent classes of labeled seeds. Note that the edges in an exchange graph lost the 'color' of labels. The exchange graph not necessary be a finite graph, if it is finite, then we say the corresponding cluster algebra (and its exchange pattern) is of *finite type*. The exchange graph is related to a cluster complex Δ of $\mathcal{A}$, which is a simplicial complex on the ground set $\mathcal{X}$ with the clusters as the maximal simplices. Then Δ is a n-dimensional complex. If $\mathcal{A}$ is of finite type or skew-symmetric, then the vertices of $E_{\mathcal{A}}$ are clusters, thus the dual graph of Δ is $E_{\mathcal{A}}$.

2.2 Finite types and finite mutation types

By the classification of cluster algebras of finite type [9], a cluster algebra is of finite type if and only if there is a seed whose quiver is one of quivers depicted in Figure 1. For a quiver mutation equivalent to a quiver in Figure 1, we call it a quiver of corresponding type. Note that the underlying graphs of quivers in Figure 1 are trees, thus any two quivers with the same underlying graph are mutation equivalent, and they have the same type.

Now we recall the classification of skew-symmetric cluster algebras of finite mutation type. Let us start with definition of block-decomposable quivers.

$$A_n : \quad 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad n\text{-}1 \longrightarrow n$$

$$B_n : \quad 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad n\text{-}1 \xrightarrow{(2,1)} n$$

$$C_n : \quad 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad n\text{-}1 \xrightarrow{(1,2)} n$$

$$D_n : \quad \begin{tikzcd} & & & & n\text{-}1 \\ 1 \arrow[r] & 2 \arrow[r] & \cdots & n\text{-}2 \arrow[ur] \arrow[dr] & \\ & & & & n \end{tikzcd}$$

$$E_6 : \quad \begin{tikzcd} & & 6 & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \end{tikzcd}$$

$$E_7 : \quad \begin{tikzcd} & & 7 & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \end{tikzcd}$$

$$E_8 : \quad \begin{tikzcd} & & 8 & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[u] \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

$$F_4 : \quad 1 \longrightarrow 2 \xrightarrow{(2,1)} 3 \longrightarrow 4$$

$$G_3 : \quad 1 \xrightarrow{(3,1)} 2$$

Figure 1: Quivers of finite type

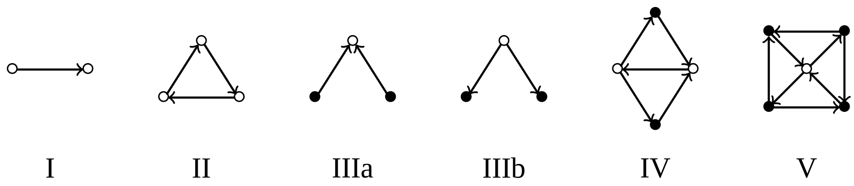

Figure 2: Blocks. Outlets are colored white, dead ends are black.

Definition 2.7. *[6, 8] A block is a quiver isomorphic to one of the quivers with black/white colored vertices shown on Figure 2. Vertices marked in white are called outlets. A connected quiver Q is called block-decomposable (decomposable for brevity) if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching (matching of outlets of the same block is not allowed), where two arrows with same endpoints and opposite directions cancel out. If Q is not block-decomposable then we call Q non-decomposable.*

Then it is proved in [6] (Theorem 13.3) that a quiver is decomposable if and only if it is a quiver of a triangulation of an oriented marked Riemann surface, and thus a quiver mutation equivalent to a decomposable quiver is also decomposable. Note that all arrow multiplicities of a decomposable quiver are 1 or 2. Therefore decomposable quivers are mutation finite. It is clearly that a quiver of rank two, that is, a quiver with two vertices, is mutation finite. Besides these two kinds of quivers, there are exactly 11 exceptional skew-symmetric quivers of finite mutation type, see Theorem 6.1 in [8]. We list the exceptional quivers in Figure 3.

2.3 Automorphism groups

In this subsection, we recall the cluster automorphism group[1] of a cluster algebra, and the automorphism group of the corresponding exchange graph[4].

Definition 2.8. *[1](Cluster automorphisms) For a cluster algebra $\mathcal{A}$ and a $\mathbb{Z}$-algebra automorphism $f : \mathcal{A} \to \mathcal{A}$, we call f a* cluster automorphism, *if there exists a labeled seed $(\mathbf{x}, B)$ of $\mathcal{A}$ such that the following conditions are satisfied:*

1. *$f(\mathbf{x})$ is a cluster;*
2. *f is compatible with mutations, that is, for every $x \in \mathbf{x}$ and $y \in \mathbf{x}$, we have*

$$f(\mu_{x,\mathbf{x}}(y)) = \mu_{f(x),f(\mathbf{x})}(f(y)).$$

Then a cluster automorphism maps a labeled seed $\Sigma = (\mathbf{x}, B)$ to a labeled seed $\Sigma' = (\mathbf{x}', B')$. Note that in a labeled seed, the cluster is a ordered set, then the second item in above definition yields that $B' = B$ or $B' = -B$. In fact, under our assumption that B is indecomposable, we have the following

Lemma 2.9. *[1] A $\mathbb{Z}$-algebra automorphism $f : \mathcal{A} \to \mathcal{A}$ is a cluster automorphism if and only if one of the following conditions is satisfied:*

1. *there exists a labeled seed $\Sigma = (\mathbf{x}, B)$ of $\mathcal{A}$, such that $f(\mathbf{x})$ is the cluster in a labeled seed $\Sigma' = (\mathbf{x}', B')$ of $\mathcal{A}$ with $B' = B$ or $B' = -B$;*

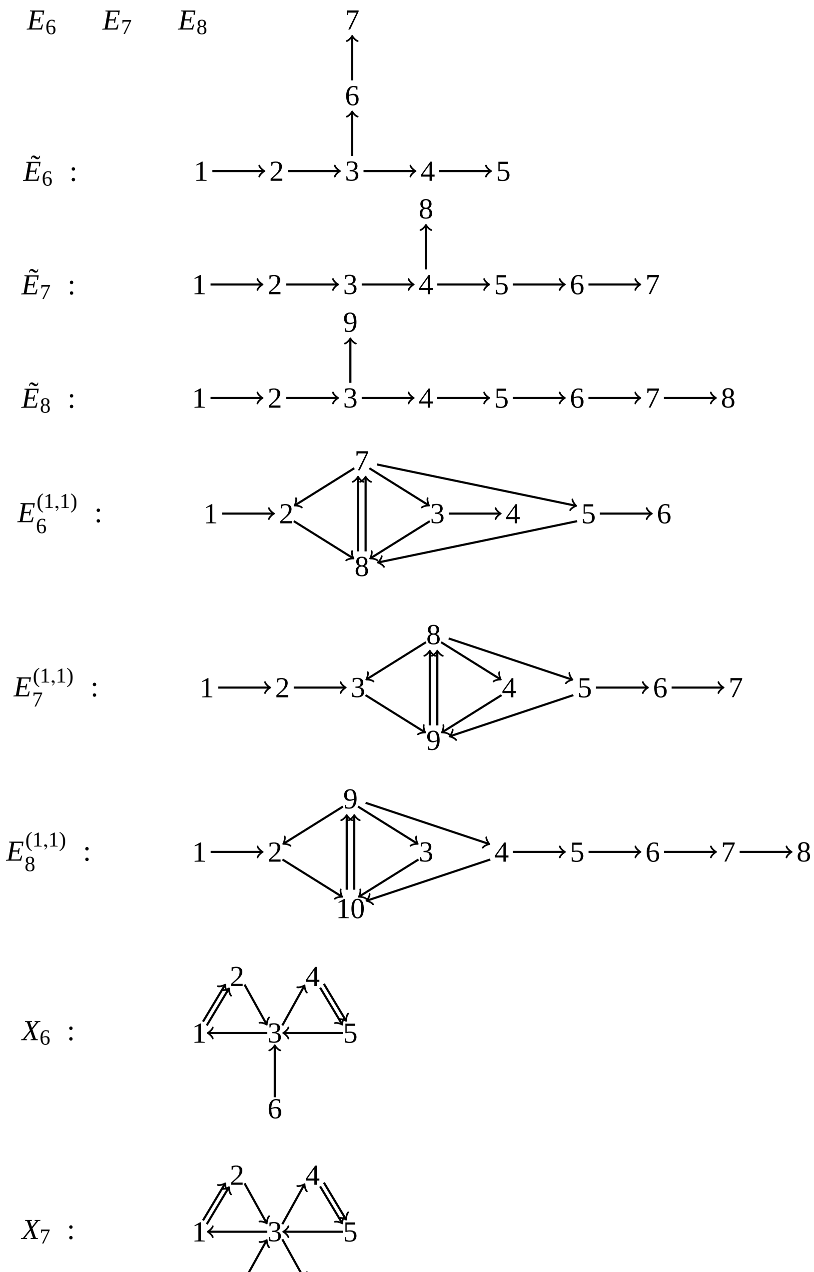

Figure 3: Non-decomposable quivers of finite mutation type

2. *for every labeled seed* $\Sigma = (\mathbf{x}, B)$ *of* $\mathcal{A}$, $f(\mathbf{x})$ *is the cluster in a labeled seed* $\Sigma' = (\mathbf{x}', B')$ *with* $B' = B$ *or* $B' = -B$.

We call those cluster automorphism such that $B = B'$ ($B = -B'$ respectively) the *direct cluster automorphism* (*inverse cluster automorphism* respectively). Clearly, all the cluster automorphism of a cluster algebra $\mathcal{A}$ consist a group with homomorphism compositions as multiplications. We call this group the *cluster automorphism group* of $\mathcal{A}$, and denote it by $Aut(\mathcal{A})$. We call the group $Aut^{+}(\mathcal{A})$ consisting of the direct cluster automorphisms of $\mathcal{A}$ the *direct cluster automorphism group* of $\mathcal{A}$, which is a subgroup of $Aut(\mathcal{A})$ with index at most two[1].

Definition 2.10. *(Automorphism of exchange graphs)[4] An automorphism of the exchange graph* $E_{\mathcal{A}}$ *of a cluster algebra* $\mathcal{A}$ *is an automorphism of* $E_{\mathcal{A}}$ *as a graph, that is, a permutation* σ *of the vertex set, such that the pair of vertices* (u, v) *forms an edge if and only if the pair* $(\sigma(u), \sigma(v))$ *also forms an edge.*

Clearly, the natural composition of two automorphisms of $E_{\mathcal{A}}$ is again an automorphism of $E_{\mathcal{A}}$. We define an *automorphism group* $Aut(E_{\mathcal{A}})$ of $E_{\mathcal{A}}$ as a group consisting of automorphisms of $E_{\mathcal{A}}$ with compositions of automorphisms as multiplications.

It is clearly that an cluster automorphism induces a unique automorphism of the exchange graph. Thus $Aut(E_{\mathcal{A}})$ is a subgroup of $Aut(\mathcal{A})$ [4]. By the definition, an automorphism σ of an exchange graph maps clusters to clusters, and induces an automorphism of its dual graph: cluster complex Δ, we denote this automorphism by σ_{Δ}. Then σ_{Δ} is a permutation of cluster variables of $\mathcal{A}$, but it may not be compatible with algebra relations of cluster variables in $\mathcal{A}$, thus it is not a cluster automorphism, and in this case $Aut(E_{\mathcal{A}})$ is a proper subgroup of $Aut(\mathcal{A})$, for example see Example 2 and Example 4. However, we have the following:

Lemma 2.11. *Let* $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}}$ *be an automorphism which maps a seed* $\Sigma = (\mathbf{x}, B)$ *to a seed* $\Sigma' = (\mathbf{x}', B')$. *If* $B \cong B'$ *or* $B \cong -B'$ *under the map* $\sigma_{\Delta} : \mathbf{x} \to \mathbf{x}'$, *then* $\sigma_{\Delta} : \mathbf{x} \to \mathbf{x}'$ *induces a cluster automorphism* δ *of* $\mathcal{A}$ *and the induced automorphism* $\delta_E : E_{\mathcal{A}} \to E_{\mathcal{A}}$ *is the same as* σ.

Proof. Since $B \cong B'$ or $B \cong -B'$ under the map $\sigma_{\Delta} : \mathbf{x} \to \mathbf{x}'$, $\sigma : \mathbf{x} \to \mathbf{x}'$ induces a cluster automorphism δ of $\mathcal{A}$. To prove $\delta_E = \sigma : E_{\mathcal{A}} \to E_{\mathcal{A}}$, we only need to show that $\delta = \sigma_{\Delta}$ on the cluster complex Δ, or equivalently, on the cluster variables of $\mathcal{A}$. This is true for the cluster $\mathbf{x}$, and the general cases can be proved by inductions on mutations. □

3 Automorphism groups of exchange graphs

In this section we consider relations between two groups $Aut(\mathcal{A})$ and $Aut(E_{\mathcal{A}})$ for a cluster algebra $\mathcal{A}$ of finite type or of skew-symmetric finite mutation type. For this, it is needed to describe $E_{\mathcal{A}}$ more precisely. Firstly we will recall the basic structures of $E_{\mathcal{A}}$ from [9], and then we introduce layers of geodesic loops on $E_{\mathcal{A}}$.

3.1 Layers of geodesic loops

Let $\Sigma = (\mathbf{x}, B)$ be a labeled seed on the exchange pattern $\mathbb{T}_n$ of $\mathcal{A}$. Let $\mathbf{x}'$ be a proper subset of $\mathbf{x}$, then $\mathbf{x}'$ is a non-maximal simplex in the cluster complex Δ. We denote by $\Delta_{\mathbf{x}'}$ the *link* of $\mathbf{x}'$,

which is the simplicial complex on the ground set $\mathscr{X}_{\mathbf{x}'} = \{\alpha \in \mathscr{X} - D : D \cup \{\alpha\} \in \Delta\}$, such that $\mathbf{x}''$ is a simplex in $\Delta_{\mathbf{x}'}$ if and only if $\mathbf{x}' \cup \mathbf{x}''$ is a simplex in Δ. Let $\Gamma_{\mathbf{x}'}$ be the dual graph of $\Delta_{\mathbf{x}'}$. We view $\Gamma_{\mathbf{x}'}$ as a subgraph of $E_{\mathcal{A}}$ whose vertices are the simplices in Δ that contains $\mathbf{x}'$. In fact $\Gamma_{\mathbf{x}'}$ is the exchange graph of a cluster algebra $\mathcal{A}_f$ defined by a seed $\Sigma_f = (\mathbf{x} \setminus \mathbf{x}', \mathbf{x}', B_f)$, which is the frozenization of Σ at $\mathbf{x}'$ (see Definition 2.25 [3]), where B_f is obtained from B by deleting the columns labeled by variables in $\mathbf{x}'$. Then elements in $\mathbf{x}'$ are coefficients of $\mathcal{A}_f$ (we refer to [9, 11] for a cluster algebra with coefficients). Let $\mathcal{A}'$ be cluster algebra defined by a seed $\Sigma' = (\mathbf{x} \setminus \mathbf{x}', B')$, where B' is obtained from B by deleting rows and columns labeled by variables in $\mathbf{x}'$. In our settings, that is, cluster algebras are of finite type or of skew-symmetric finite type, the exchange graph of a cluster algebra (with coefficients) only depends on the principal part of the exchange matrix [10, 2] which is the submatrix labeled by $\mathbf{x} \setminus \mathbf{x}' \times \mathbf{x} \setminus \mathbf{x}'$, thus the graph $\Gamma_{\mathbf{x}'}$ is the same as the exchange graph $E_{\mathcal{A}'}$.

For a $n-2$-dimensional subcomplex $\mathbf{x}'$ of Δ, we call the dual graph $\Gamma_{\mathbf{x}'}$ a *geodesic loop* of $E_{\mathcal{A}}$. If $\mathcal{A}$ is of finite type, then $E_{\mathcal{A}}$ is a finite graph, and $\Gamma_{\mathbf{x}'}$ is a polygon. Notice that in the seed $\Sigma' = (\mathbf{x} \setminus \mathbf{x}', B')$, B' is of Dynkin type, that is, one of types A_2, B_2, C_2 or G_2. Therefore $\Gamma_{\mathbf{x}'}$ is a $h+2$-polygon, where h is the Coexter number of the corresponding Dynkin type[10]. If $\mathcal{A}$ is of finite mutation type, then $\Gamma_{\mathbf{x}'}$ may be a line.

In fact, the fundamental group of $E_{\mathcal{A}}$ is generated by geodesic loops pinned down to a fixed basepoint, that is, the generators of $E_{\mathcal{A}}$ is of the form $PL\overline{P}$, where P is a path originating at the basepoint, $\overline{P}$ is the inverse path of P, and L is a geodesic loop. We fix a basepoint $\Sigma = (\mathbf{x}, B)$ and introduce the following concept.

Definition 3.1. 1. *Let Σ' be a point of $E_{\mathcal{A}}$, the distance $\ell(\Sigma, \Sigma')$ between Σ and Σ' is the minimal length of paths between Σ and Σ';*

2. *Let L be a geodesic loop of $E_{\mathcal{A}}$, the distance $\ell_{\Sigma}(L)$ between Σ and L is equal to the minimal length $min\{\ell(\Sigma, \Sigma'), \Sigma' \in L\}$;*

3. *Let $m \in \mathbb{Z}_{\geqslant 0}$ be a non-negative integer, denote by ℓ_{Σ}^{m} the set of geodesic loop whose distance to Σ is m. We call it the m-layer of geodesic loops of $E_{\mathcal{A}}$ originating at Σ;*

4. *For a set ℓ_{Σ}^{m}, the element in $N(\ell_{\Sigma}^{m})$ is the number of edges of geodesic loops in ℓ_{Σ}^{m}.*

Remark 3.2. *The following observations are directly derived from the definitions:*

1. *The elements in ℓ_{Σ}^{0} are those geodesic loops $\Gamma_{\mathbf{x}'}$ for the $n-2$-dimensional subcomplex $\mathbf{x}'$ of Δ;*

2. *For $m_1 \neq m_2$, $\ell_{\Sigma}^{m_1} \cap \ell_{\Sigma}^{m_2} = \emptyset$;*

3. *The disjoint union $\sqcup_{m \geqslant 0} \ell_{\Sigma}^{m}$ is the set of all the geodesic loops of $E_{\mathcal{A}}$;*

4. *If $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ is an isomorphism of graphs, such that the image of Σ is Σ', then for every $m \in \mathbb{Z}_{\geqslant 0}$, $N(\ell_{\Sigma}^{m}) = N(\ell_{\Sigma'}^{m})$.*

3.2 Cases of rank two and rank three

In this subsection, we consider the relations between $Aut(\mathcal{A})$ and $Aut(E_{\mathcal{A}})$ for a cluster algebra $\mathcal{A}$ of rank two or rank three.

Example 2. *For a finite type cluster algebra $\mathcal{A}$ of rank 2, that is, one of types A_2, B_2, C_2 or G_2, its exchange graph $E_{\mathcal{A}}$ is a $h+2$-polygon, thus $Aut(E_{\mathcal{A}}) \cong D_{h+2}$. If $\mathcal{A}$ is of type A_2, then $Aut(\mathcal{A}) \cong D_5$ [1], thus $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$. If $\mathcal{A}$ is of type B_2, C_2 or G_2, Theorem 3.5 in [5] shows that $Aut(\mathcal{A}) \cong D_{h+2/2}$, thus $Aut(\mathcal{A}) \subsetneqq Aut(E_{\mathcal{A}})$.*

Example 3. *For an infinite type skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is a line, thus $Aut(E_{\mathcal{A}}) =< s > \times < r > \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where s is a left shift of $E_{\mathcal{A}}$ which maps a cluster to the left adjacent cluster and r is a refection through a fixed cluster. Then s corresponds to a direct cluster automorphism of $\mathcal{A}$ and r corresponds to an inverse cluster automorphism of $\mathcal{A}$, thus by Lemma 2.11 $Aut(E_{\mathcal{A}}) \subseteq Aut(\mathcal{A})$. Therefore $Aut(E_{\mathcal{A}}) \cong Aut(\mathcal{A}) \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.*

Example 4. *For an infinite type non-skew-symmetric cluster algebra $\mathcal{A}$ of rank 2, its exchange graph $E_{\mathcal{A}}$ is also a line, thus as showed in Example 3, $Aut(E_{\mathcal{A}}) =< s > \times < r > \cong \mathbb{Z} \rtimes \mathbb{Z}_2$, where s corresponds to a direct cluster automorphism of $\mathcal{A}$, while r dose not correspond to any cluster automorphism of $\mathcal{A}$, since there is no non-trivial symmetry of the quiver in any seed of $\mathcal{A}$. Thus $Aut(\mathcal{A}) \cong \mathbb{Z} \subsetneqq Aut(E_{\mathcal{A}})$.*

Example 5. *We consider the cluster algebra $\mathcal{A}$ of type A_3 with an initial labeled seed $\Sigma = (\{x_1, x_2, x_3\}, Q)$, where Q is $1 \longrightarrow 2 \longleftarrow 3$. Then its exchange graph $E_{\mathcal{A}}$ is depicted in Figure 4. Note that there are three quadrilaterals and six pentagons in $E_{\mathcal{A}}$. Then as showed in [5] (Example 2), $Aut(\mathcal{A}) =< f_-, f_0 > \cong D_6$, where f_- is defined by:*

$$f_- : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto \mu_2(x_2) \\ x_3 \mapsto x_3 \end{cases} \tag{3}$$

It maps Σ to Σ_1, and induces a reflection with respect to the horizontal central axis of $E_{\mathcal{A}}$, see in Figure 4. The cluster automorphism f_0 is defined by:

$$f_0 : \begin{cases} x_1 \mapsto x_3 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_1 \end{cases} \tag{4}$$

It induces a reflection with respect to the vertical central axis of $E_{\mathcal{A}}$. In fact as showed in [5], an cluster automorphism of $\mathcal{A}$ induces a permutation of seeds in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$, thus $Aut(\mathcal{A}) \cong D_6$ can be viewed as the symmetry group of the belt consisting of $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$. We prove that these permutations induce all the automorphisms of $E_{\mathcal{A}}$, that is, $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$. For this purpose, by Lemma 2.11, we only need to show that there exists no automorphism of $E_{\mathcal{A}}$ which maps Σ to a vertex excepting for a vertex in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5\}$. Let σ be an automorphism of $E_{\mathcal{A}}$, then due to symmetries of $E_{\mathcal{A}}$, we only show that $\sigma(\Sigma) \neq O_i$, $i = 1, 2, 3$. The layers

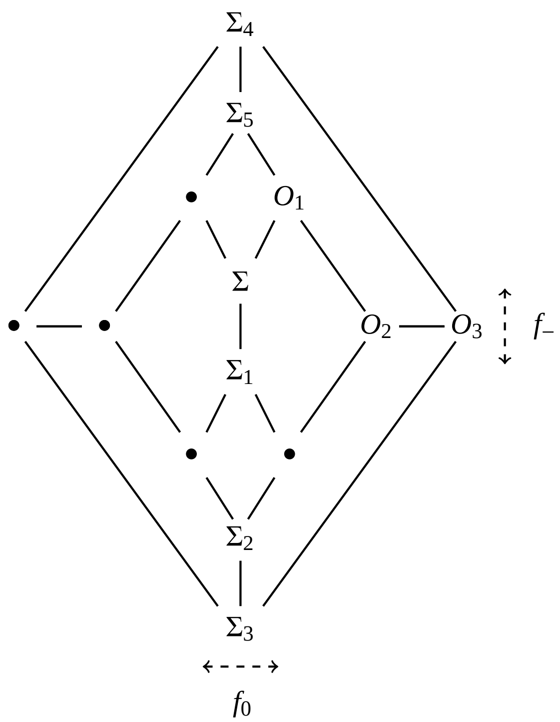

Figure 4: The exchange graph of a cluster algebra of type A_3

of geodesic loops originating at these vertices are as follows:

$$N(\ell^0_\Sigma) = \{4,5,5\},\ N(\ell^1_\Sigma) = \{4,5,5\},\ N(\ell^2_\Sigma) = \{5,5\},\ N(\ell^3_\Sigma) = \{4\};$$
$$N(\ell^0_{O_1}) = \{4,5,5\},\ N(\ell^1_{O_1}) = \{5,5,5\},\ N(\ell^2_{O_1}) = \{4,4\},\ N(\ell^3_{O_1}) = \{5\};$$
$$N(\ell^0_{O_2}) = \{5,5,5\},\ N(\ell^1_{O_1}) = \{4,4,4\},\ N(\ell^2_{O_1}) = \{5,5,5\};$$
$$N(\ell^0_{O_3}) = \{4,5,5\},\ N(\ell^1_{O_3}) = \{5,5,5\},\ N(\ell^2_{O_3}) = \{4,4\},\ N(\ell^3_{O_3}) = \{5\};$$

Then by Remark 3.2(4), $\sigma(\Sigma) \neq O_i$, $i = 1,2,3$. Thus $Aut(E_{\mathcal{A}}) \cong Aut(\mathcal{A}) \cong D_6$.

Example 6. *It is well known that the cluster algebras of type B_n and type C_n have the same combinatorial structure. Originating at a seed Σ, the exchange graph of a cluster algebra $\mathcal{A}$ of type B_3 or type C_3 is depicted in Figure 5. For the cluster algebra of type B_3, the quiver of the initial seed Σ is*

$$1 \longrightarrow 2 \xleftarrow{(2,1)} 3.$$

For the cluster algebra of type C_3, the quiver of the initial seed Σ is

$$1 \longrightarrow 2 \xleftarrow{(1,2)} 3.$$

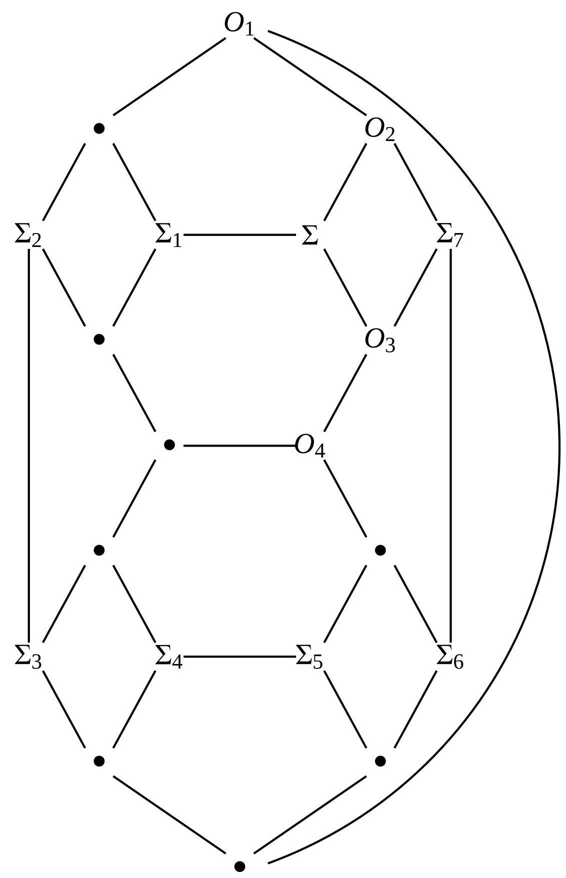

Figure 5: The exchange graph of a cluster algebra of type B_3 or type C_3

Let σ be an automorphism of $E_{\mathcal{A}}$. Since $N(\ell_{\Sigma}^{0}) = \{4,5,6\}$, there are no rotation symmetries of $E_{\mathcal{A}}$ at Σ, thus $\sigma(\Sigma) \neq \Sigma$. As showed by Example 3 in [5], Aut$(\mathcal{A}) \cong D_4$ and $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$ are all the seeds which quivers are isomorphic to Q, then by Lemma 2.11, to prove that Aut$(E_{\mathcal{A}}) \cong$ Aut$(\mathcal{A})$, we only need to show that $\sigma(\Sigma)$ must be a seed in $\{\Sigma, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6, \Sigma_7\}$. By the symmetries of $E_{\mathcal{A}}$, we only prove that $\sigma(\Sigma) \neq O_i (i = 1,2,3,4)$, and this can be obtained by the fact that these seeds have different combinatorial numbers of layers of geodesic loops:

$$N(\ell_{\Sigma}^{0}) = \{4,5,6\},\ N(\ell_{\Sigma}^{1}) = \{4,5,6\};$$
$$N(\ell_{O_1}^{0}) = \{5,6,6\};$$
$$N(\ell_{O_2}^{0}) = \{4,5,6\},\ N(\ell_{O_2}^{1}) = \{5,6,6\};$$
$$N(\ell_{O_3}^{0}) = \{4,5,6\},\ N(\ell_{O_3}^{1}) = \{5,6,6\};$$
$$N(\ell_{O_4}^{0}) = \{5,6,6\}.$$

Example 7. *For cluster algebra of type F_4, let the quiver Q of a seed Σ is*

$$1 \longrightarrow 2 \xleftarrow{(2,1)} 3 \longrightarrow 4.$$

Then $Aut(\mathcal{A}) \cong D_7$ [5]. The variables x_1, x_2, x_3 and the corresponding full subquiver of Q form a seed Σ_1 of type B_3, while x_2, x_3, x_4 and the corresponding full subquiver of Q form a seed Σ_2 of type C_3. By pinning down Σ, rotating the graph $E_{\mathcal{A}}$ induces an automorphism σ of $E_{\mathcal{A}}$, which exchanges the graph $E_{\mathcal{A}_{\Sigma_1}}$ and the graph $E_{\mathcal{A}_{\Sigma_2}}$. However σ does not induces a cluster automorphism of $\mathcal{A}$, and $Aut(\mathcal{A}) \cong D_7 \subsetneq D_7 \rtimes \mathbb{Z}_2 \cong Aut(E_{\mathcal{A}})$.

Proposition 3.3. *Let Q be a connected quiver with three vertices, which is of finite type. Let $\Sigma = (\mathbf{x}, Q)$ and $\Sigma' = (\mathbf{x}', Q')$ be two seeds. If there is an isomorphism $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ such that $\sigma(\Sigma) = \Sigma'$, then $\Sigma' = (\mathbf{x}', Q')$ is a finite type seed with Q' connected, and*

1. *if Σ is of type A_3, then $Q' \cong Q$ (or Q^{op});*
2. *if Σ is of type B_3 and Σ' is not of type C_3, then $Q' \cong Q$ (or Q^{op});*
3. *if Σ is of type C_3 and Σ' is not of type B_3, then $Q' \cong Q$ (or Q^{op}).*

Proof. Clearly, since $E'_{\mathcal{A}} \cong E_{\mathcal{A}}$ is of finite, Q' is a Dynkin type quiver with three vertices. If Q is of type A_3, then by Example 5,

$$N(\ell^0_{\Sigma}) = \{4, 5, 5\} \text{ or } \{5, 5, 5\}.$$

If Q is of type B_3 (or C_3), then from Example 6,

$$N(\ell^0_{\Sigma}) = \{4, 5, 6\} \text{ or } \{5, 6, 6\}.$$

If Q' is a union of a quiver of type A_2 and a point, then from Example 2,

$$N(\ell^0_{\Sigma}) = \{4, 4, 5\}.$$

If Q' is a union of a quiver of type B_2 (or C_2) and a point, then from Example 2,

$$N(\ell^0_{\Sigma}) = \{4, 4, 6\}.$$

If Q' is a union of a quiver of type G_2 and a point, then from Example 2,

$$N(\ell^0_{\Sigma}) = \{4, 4, 8\}.$$

Thus we get the proof by Remark 3.2. □

Example 8. *Let Q be the quiver in Figure 6, we call it of type $\tilde{A}_2$. Then it is not hard to see that if a quiver in the mutation class of Q is not isomorphic to Q, then it must be isomorphic to the quiver Q' in Figure 6. Let $\mathcal{A}$ be a cluster algebra with an initial seed $\Sigma = (\{x_1, x_2, x_3\}, Q)$, similar to above examples, to show that $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$, it is only need to notice that:*

$$N(\ell^0_{\Sigma}) = \{5, 5, \infty\},$$
$$N(\ell^0_{\Sigma'}) = \{5, 5, 5\},$$

Figure 6: quivers of type $\tilde{A}_2$

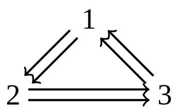

Figure 7: quiver of type T_3

where Σ' is a seed of $\mathcal{A}$ with quiver isomorphic to Q'. In fact, from section 3.3 in [1], $Aut(\mathcal{A}) =< r_1, r_2 | r_1 r_2 = r_2 r_1, {r_1}^2 = r_2 > \rtimes < \sigma | \sigma^2 = 1 > \cong H_{2,1} \rtimes \mathbb{Z}_2$, where

$$r_1 : \begin{cases} x_1 \mapsto x_3 \\ x_2 \mapsto \mu_1(x_1) \\ x_3 \mapsto x_2 \end{cases} \tag{5}$$

$$r_2 : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto \mu_3\mu_1(x_3) \\ x_3 \mapsto \mu_1(x_1) \end{cases} \tag{6}$$

$$\sigma : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto x_1 \\ x_3 \mapsto x_3 \end{cases} \tag{7}$$

Thus $Aut(E_{\mathcal{A}}) \cong H_{2,1} \rtimes \mathbb{Z}_2$

Example 9. *Let $\mathcal{A}$ be a cluster algebra from an once punctured torus, we call it a cluster algebra of type T_3, then it is of finite mutation type with quiver always isomorphic to the quiver in Figure 7. Then by Lemma 2.11, we have $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$.*

Lemma 3.4. *Let Q be a connected skew-symmetric quiver of finite mutation type.*

1. *If there are 3 vertices in Q, then Q is one of the following types:*
   - *(1) A_3 type;*
   - *(2) $\tilde{A}_2$ type;*
   - *(3) T_3 type.*

2. *If there are at least* 4 *vertices in* Q*, then any full subquiver of* Q *with three vertices is of type* A_3 *or of type* $\tilde{A}_2$*.*

Proof. 1. From the classification of cluster algebras of finite mutation type, Q must be block-decomposable, then the proof is a straightforward check by gluing the blocks in 2.

2. It is only need to notice that there is no way to glue a quiver of type T_3 with other blocks. □

It is clearly that if for any quiver in the mutation equivalent class of Q, the number of arrows between any two vertices is at most 2, then Q is of finite mutation type. The above lemma shows that the inverse statement is also true for the cases of vertices at least 3, that is, we have the following:

Corollary 3.5. *A connected quiver* Q *with at least 3 vertices is of finite mutation type if and only if for any quiver in its mutation equivalent class, the number of arrows between any two vertices is at most* 2*.*

Proposition 3.6. *Let* Q *be a connected skew-symmetric quiver with three vertices, which is of finite mutation type. Let* $\Sigma = (\mathbf{x}, Q)$ *and* $\Sigma' = (\mathbf{x}', Q')$ *be two seeds. If there is an isomorphism* $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ *such that* $\sigma(\Sigma) = \Sigma'$*, then* $Q' \cong Q$ *or* $Q' \cong Q^{op}$*.*

Proof. Similar to Proposition 3.3, this follows from Lemma 3.4, Example 9, Example 5 and Example 8. □

3.3 General cases

Theorem 3.7. *Let* $\mathcal{A}$ *be a cluster algebra of finite type. Assume that it is not of type* F_4*, let* $\Sigma = (\mathbf{x}, Q)$ *be a labeled seed of* $\mathcal{A}$*, where* Q *is a connected quiver with at least three vertices. Then we have* $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$*.*

Proof. We need to show that $Aut(E_{\mathcal{A}}) \subseteq Aut(\mathcal{A})$. Let $\mathbf{x}' \in \mathbf{x}$ be a 3-dimensional complex, such that the full subquiver Q' of Q corresponding to the variables in $\mathbf{x}'$ are connected. Define a seed $\Sigma' = (\mathbf{x}', Q')$. Denote by $\mathcal{A}'$ the cluster algebra defined by Σ'. Let σ' be an automorphism of $E_{\mathcal{A}}$. Then σ' induces an automorphism σ of Δ, which maps $\Delta_{\mathbf{x}\backslash\mathbf{x}'}$ to $\Delta_{\sigma(\mathbf{x}\backslash\mathbf{x}')}$. Clearly, $\sigma : \Delta_{\mathbf{x}\backslash\mathbf{x}'} \to \Delta_{\sigma(\mathbf{x}\backslash\mathbf{x}')}$ is an isomorphism, and induces an isomorphism from $\Gamma_{\mathbf{x}\backslash\mathbf{x}'}$ to $\Gamma_{\sigma(\mathbf{x}\backslash\mathbf{x}')}$. Let $\Sigma'' = (\mathbf{x}'', Q'')$ be a seed, where $\mathbf{x}'' = \sigma(\mathbf{x}')$ and Q'' is a full subquiver of $Q(\mathbf{x})$ whose vertices are those labeled by elements in $\mathbf{x}''$. Let $\mathcal{A}''$ be the cluster algebra of Σ''. Then as showed in the beginning of subsection 3.1, $\Gamma_{\mathbf{x}\backslash\mathbf{x}'} \cong E_{\mathcal{A}'}$ and $\Gamma_{\sigma(\mathbf{x}\backslash\mathbf{x}')} \cong E_{\mathcal{A}''}$. Thus $E_{\mathcal{A}'} \cong E_{\mathcal{A}''}$. Since $\mathcal{A}$ is not of type F_4, if Q' is of type B_3 (type C_3 respectively), then Q'' is not of type C_3 (type B_3 respectively). Thus by Proposition 3.3, $Q' \cong Q''$ or $Q' \cong Q''^{op}$. Finally, due to the arbitrariness of the choose of $\mathbf{x}'$ and the connectivity of Q, $Q(\sigma(\mathbf{x})) \cong Q$ or $Q(\sigma(\mathbf{x})) \cong Q^{op}$. Therefore $\sigma : \Delta \to \Delta$ induces an cluster automorphism of $\mathcal{A}$. Thus $Aut(E_{\mathcal{A}}) \subseteq Aut(\mathcal{A})$ and $Aut(E_{\mathcal{A}}) \cong Aut(\mathcal{A})$. □

Then combine with above theorem, Table 1 in [1] and Theorem 3.5 in [5], we have the table 1 of automorphism groups of the exchange graphs of cluster algebras of finite type. The cases of rank two and type F_4 is computed in Example 2 and Example 7 respectively.

Theorem 3.8. *Let* $\mathcal{A}$ *be a connected skew-symmetric cluster algebra of finite mutation type, then* $Aut(\mathcal{A}) \cong Aut(E_{\mathcal{A}})$*.*

Dynkin type	Automorphism group $Aut(E_{\mathcal{A}})$
$A_n(n \geqslant 2)$	D_{n+3}
B_2	D_6
$B_n(n \geqslant 3)$	D_{n+1}
C_2	D_6
$C_n(n \geqslant 3)$	D_{n+1}
D_4	$D_4 \times S_3$
$D_n(n \geqslant 5)$	$\mathbb{Z}_2$
E_6	D_{14}
E_7	D_{10}
E_8	D_{16}
F_4	$D_7 \rtimes \mathbb{Z}_2$
G_2	D_8

Table 1: Automorphism groups of exchange graphs of cluster algebras of finite type

Proof. If $\mathcal{A}$ is of finite type of rank 2, that is, of type A_2, then the result follows from Example 2. If $\mathcal{A}$ is of infinite type of rank 2, then the result follows from Example 3. For the case of type A_3, type $\tilde{A}_2$ or type T_3, we derive the result from Example 5, Example 8 or Example 9 respectively, otherwise, by Lemma 3.4(2), the connected full subquiver of a quiver of $\mathcal{A}$ with three vertices is of type A_3 or of type $\tilde{A}_2$. Therefore by Example 8 and Example 5, the proof is similar to the proof of Theorem 3.7. □

Corollary 3.9. *Let $\mathcal{A}$ be a connected cluster algebra of finite type or of skew-symmetric finite mutation type, then an automorphism of $E_{\mathcal{A}}$ is determined by the image of any fixed seed and the images of seeds which are adjacent to the fixed seed, more precisely, let $\Sigma = (\mathbf{x}, B)$ be a seed on $E_{\mathcal{A}}$, then an automorphism $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}}$ is determined by a pair (Σ', σ'), where $\Sigma' = (\mathbf{x}', B')$ is a seed on $E_{\mathcal{A}}$ and $\sigma' : \mathbf{x} \to \mathbf{x}'$ is a bijection such that $\sigma(\Sigma) = \Sigma'$ and $\sigma(\mu_x(\mathbf{x})) = \mu_{\sigma'(x)}(\mathbf{x}')$ for any $x \in \mathbf{x}$.*

Proof. If $\mathcal{A}$ is of finite type of rank 2 and type of F_4, then the conclusion is clearly. Otherwise, note that a cluster automorphism is determined by such a pair (Σ', σ'), thus the proof follows from Theorem 3.7 and Theorem 3.8. □

Conjecture 3.10. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two connected cluster algebras of finite type, or of skew-symmetric finite mutation type. Let $\Sigma = (\mathbf{x}, B)$ and $\Sigma' = (\mathbf{x}', B')$ be two seeds of $\mathcal{A}$ and $\mathcal{A}'$ respectively.*

1. *If $N(\ell^k_{\Sigma}) = N(\ell^k_{\Sigma'})$ for any $k \in \mathbb{Z}^*$, then there exists an isomorphism $\sigma : E_{\mathcal{A}} \to E_{\mathcal{A}'}$ such that $\sigma(\Sigma) = \Sigma'$;*

2. *Assume that $\mathcal{A}$ or $\mathcal{A}'$ is not types of rank 2 and type F_4. If $N(\ell^k_{\Sigma}) = N(\ell^k_{\Sigma'})$ for any $k \in \mathbb{Z}^*$, then $B \cong B'$ or $B \cong -B'$.*

It follows from Example 5, Example 6, Example 8 and Example 9 that the conjecture is true for the cases of rank 2 and rank 3. Clearly, the second part of the conjecture can be derived from the first one, Theorem 3.7 and Theorem 3.8.

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